SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2003
TRIAL HIGHER SCHOOL CERTIFICATE

Mathematics Extension 2

General Instructions

- Reading Time – 5 Minutes
- Working time – 3 hours
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.

Total Marks – 120

- Attempt all questions.
- All questions are of equal value.
- Each section is to be answered in a separate bundle, labeled Section A (Questions 1, 2, 3), Section B (Questions 4, 5, 6) and Section C (Questions 7 and 8).

Examiner: *C.Kourtesis*

Note: *This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.*

Section A
Start a new answer sheet

Question 1. (Start a new answer sheet.) (15 marks)

Marks

(a) Find $\int \frac{dx}{\sqrt{4-9x^2}}$. **2**

(b) Find $\int \frac{4}{(x-1)(2-x)}\,dx$ **3**

(c) Use integration by parts to find **3**

$$\int te^{\frac{t}{4}}\,dt$$

(d) Use the substitution $u = 2+\cos\theta$ to show that **4**

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{2+\cos\theta}\,d\theta = 2 + 4\log_e\left(\frac{2}{3}\right)$$

(e) Evaluate $\int_0^{2\pi} |\sin x|\,dx$ **2**

(f) Determine whether the following statement is True of False, and give a brief reason for your answer. **1**

$$\int_{-1}^{4} \frac{dx}{x^3} = \frac{15}{32}$$

Question 2. (15 marks)

Marks

(a) (i) Express $w = -1 - i$ in modulus-argument form. **2**

(ii) Hence express w^{12} in the form $x + iy$ where x and y are real numbers. **2**

(b) Find the equation, in Cartesian form, of the locus of the point z if **2**

$$|z - i| = |z + 3|.$$

(c) Sketch the region in the Argand diagram that satisfies the inequality **3**

$$\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}.$$

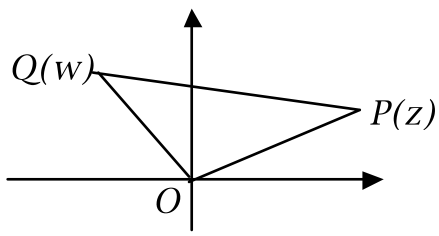
(d) (i) On the Argand diagram draw a neat sketch of the locus specified by **1**

$$\arg(z+1) = \frac{\pi}{3}.$$

(ii) Hence find z so that $|z|$ is a minimum. **2**

(e) Points P and Q represent the complex numbers z and w respectively in the Argand Diagram.

If ΔOPQ (where O is the origin) is equilateral



(i) Explain why $wz = z^2 cis\frac{\pi}{3}$. **1**

(ii) Prove that $z^2 + w^2 = zw$. **2**

Question 3. (15 marks)

Marks

(a) Sketch the following curves on separate diagrams, for $-\frac{3\pi}{2} \le x \le \frac{3\pi}{2}$.
[Note: There is no need to use calculus.]

(i) $y = \tan x$ **1**

(ii) $y = |\tan x|$ **1**

(iii) $y = \tan|x|$ **1**

(iv) $y = \tan^2 x$ **2**

(b) Consider the function $f(x) = \frac{x}{\ln x}$, $x > 0$

(i) Determine the domain and write down the equations of any asymptotes. **2**

(ii) Show that there is a minimum turning point at (e, e). **3**

(iii) Show that there is a point of inflexion at $x = e^2$. **3**

(iv) Sketch the graph of $y = f(x)$. **2**

Section B
Start a new booklet.

Question 4 (15 marks)

Marks

(a) (i) By solving the equation $z^3 = 1$ find the three cube roots of 1. **2**

(ii) Let w be a cube root of 1 where w is not real. Show that $1 + w + w^2 = 0$. **1**

(iii) Find the quadratic equation, with integer coefficients, that has roots $4 + w$ and $4 + w^2$. **3**

(b) A monic cubic polynomial, when divided by $x^2 + 4$ leaves a remainder of $x + 8$ and when divided by x leaves a remainder of -4. Find the polynomial in expanded form. **3**

(c) Consider the polynomial $P(z) = z^3 + az^2 + bz + c$ where a, b and c are all real.

If $P(\theta i) = 0$ where θ is real and non-zero:

(i) Explain why $P(-\theta i) = 0$ **1**

(ii) Show that $P(z)$ has one real zero. **1**

(iii) Hence show that $c = ab$, where $b > 0$. **4**

Question 5 (15 marks)

Marks

(a) A particle of mass m falls vertically from rest at a height of H metres above the Earth's surface, against a resistance mkv when its speed is v m/s. (k is a positive constant).
Let x m be the distance the particle has fallen, and v m/s its speed at x. Let g m/s^2 be the acceleration due to gravity.

(i) Show that the equation of motion is given by **1**

$$v\frac{dv}{dx} = g - kv$$

(ii) If the particle reaches the surfaced of the Earth with speed V_0, show that **4**

$$\ln\left(1 - \frac{kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0.$$

(iii) Show that the time T taken to reach the Earth's surface is given by **3**

$$T = \frac{1}{k}\ln\left(\frac{g}{g - kV_o}\right).$$

(iv) Show that $V_0 = Tg - kH$. **2**

(v) Hence prove that $T < \dfrac{1}{k} + \dfrac{kH}{g}$. **1**

(b) The letters A, B, C, D, E, F, I, O are arranged in a circle. In how many ways can this be done if at least two of the vowels are together? **2**

(c) A man has five friends. In how many ways can he invite one or more of them to dinner? **2**

Question 6 (15 marks)

Marks

(a) (i) Expand $(\cos\theta + i\sin\theta)^3$ and hence express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ respectively. **2**

(ii) Show that $\cot 3\theta = \dfrac{t^3 - 3t}{3t^2 - 1}$ where $t = \cot\theta$. **2**

(iii) Solve $\cot 3\theta = 1$ for $0 \le \theta \le 2\pi$. **2**

(iv) Hence show that $\cot\dfrac{\pi}{12}.\cot\dfrac{5\pi}{12}.\cot\dfrac{9\pi}{12} = -1$. **2**

(v) Write down a cubic equation with roots $\tan\dfrac{\pi}{12}$, $\tan\dfrac{5\pi}{12}$, $\tan\dfrac{9\pi}{12}$. **1**

[Express your answer as a polynomial equation with integer coefficients.]

(b) (i) Draw a sketch showing that if $f(x)$ and $g(x)$ are continuous functions and $f(x) > g(x) > 0$ for $a \le x \le b$ then **2**

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx.$$

(ii) Show that $y = \tan x$ is an increasing function for $\dfrac{\pi}{4} \le x \le \dfrac{\pi}{3}$. **1**

(iii) Prove that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}\,dx > \log_e\left(\frac{4}{3}\right)$. **3**

Section C
Start a new booklet

Question 7 (15 marks)

Marks

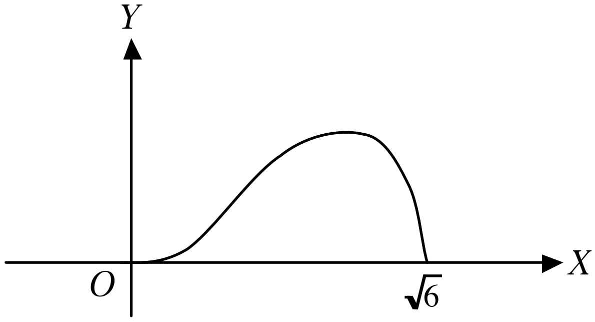
(a)

(i) If $I_n = \int_1^e x(\ln x)^n \, dx$ (where n is a non-negative integer) **3**

show that $I_n = \dfrac{e^2}{2} - \dfrac{n}{2} I_{n-1}$ (where $n \geq 1$).

(ii) Hence evaluate I_3. **2**

(b) **4**

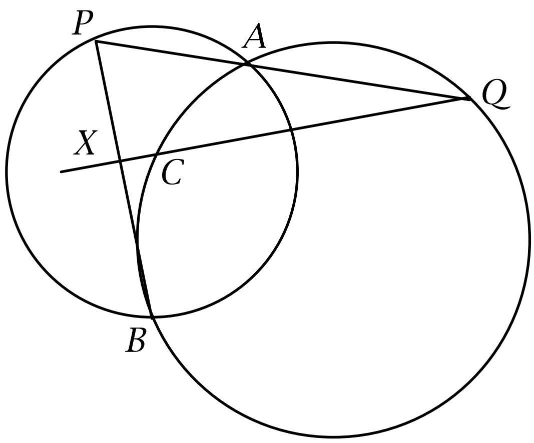


The diagram shows the graph of $y = x^2(6 - x^2)$ for $0 \leq x \leq \sqrt{6}$. The area bounded by this curve and the x-axis is rotated through one revolution about the y-axis.

Use the method of cylindrical shells to find the volume of the solid that is generated.

Question continued .....

(c)



The two circles intersect at A and B. The larger circle passes through the centre C of the smaller circle. P and Q are points on the circles such that PQ passes through A. QC is produced to meet PB at X.

Let $\angle QAB = \theta$.

(i) Make a neat copy of the diagram on your answer sheet.

(ii) Show that $\angle BCX = 180^0 - \theta$. **2**

(iii) Prove that $\angle PXC = 90^0$. **4**

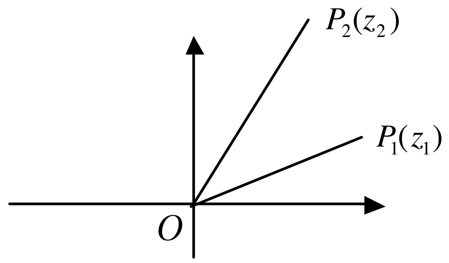
Question 8 (15 marks)

Marks

(a) Two of the roots of $x^3 + ax^2 + bx + c = 0$ are α and β. **4**

Prove that $\alpha\beta$ is a root of $x^3 - bx^2 + acx - c^2 = 0$.

(b) The points P_1 and P_2 represent the complex numbers z_1 and z_2 on the Argand diagram.



(i) Prove that $|z_1 - z_2| \geq |z_1| - |z_2|$ **2**

(ii) If $\left|z - \frac{4}{z}\right| = 2$ prove that the maximum value of $|z|$ is $\sqrt{5} + 1$. **3**

(c) (i) Prove that if the polynomial equation $P(x) = 0$ has a root of multiplicity n, then the derived polynomial equation $P'(x) = 0$ has the same root with multiplicity $n - 1$. **2**

(ii) If the equation $x^3 + 3px^2 + 3qx + r = 0$ has a repeated root, show that this root is $\frac{r - pq}{2(p^2 - q)}$, where $p^2 \neq q$. **4**

This is the end of the paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

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Mathematics Extension 2

Sample Solutions

Question 1.

(a) $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\frac{4}{9}-x^2}}$ (2)

$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + c$

(b) Let $\frac{4}{(x-1)(2-x)} = \frac{A}{x-1} + \frac{B}{2-x}$

$\therefore 4 = A(2-x) + B(x-1)$

If $x=2$: $4 = B$

If $x=1$: $4 = A$. (3)

$\therefore \int \frac{4\,dx}{(x-1)(2-x)} = 4\left(\int \frac{dx}{x-1} - \int \frac{dx}{x-2}\right)$

$= 4\left(\ln(x-1) - \ln(x-2)\right) + c$

$= 4\ln\left(\frac{x-1}{x-2}\right) + c$

(c) $\int t e^{\frac{t}{4}}\,dt$ $\quad u = t \quad v = 4e^{t/4}$

$u' = 1 \quad v' = e^{t/4}$

$= 4te^{t/4} - \int 4e^{t/4}\,dt$ (3)

$= 4te^{t/4} - 16e^{t/4} + c$

(d) $\int_0^{\pi/2} \frac{\sin 2\theta}{2+\cos\theta}\,d\theta$

Let $u = 2+\cos\theta$

$= \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{2+\cos\theta}\,d\theta$ $\quad du = -\sin\theta\,d\theta$

If $\theta = 0$ $u = 3$

$= 2\int_3^2 \frac{u-2}{u}(-du)$ $\quad$ If $\theta = \pi/2$ $u = 2$

$= 2\int_2^3 \left(1 - \frac{2}{u}\right)du$ (4)

$= 2\left[u - 2\ln u\right]_2^3$

$= 2\left\{[3 - 2\ln 3] - [2 - 2\ln 2]\right\}$

$= 2\left\{1 + 2\ln 2 - 2\ln 3\right\}$

$= 2\left\{1 + 2\ln \frac{2}{3}\right\}$

$= 2 + 4\ln \frac{2}{3}$

(e) $\int_0^{2\pi} |\sin x|\,dx$

$= 2\int_0^{\pi} \sin x\,dx$

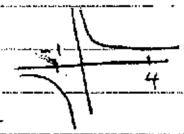
$= 2\left[-\cos x\right]_0^{\pi}$

$= 2\left\{[-(-1)] - [-1]\right\}$ (2)

$= 2 \times 2$

$= 4$

(f) $\int_{-4}^{4} \frac{dx}{x^3}$

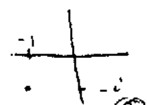


Integral is not defined due to discontinuity at $x = 0$. (1)

Question 2:

(a) (i) $w = -1 - i$

$= \sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)$



(2)

(ii) $w^{12} = \left(\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)\right)^{12}$

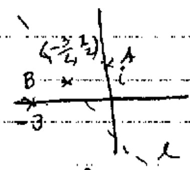
$= 2^6\ \text{cis}\left(-\frac{36\pi}{4}\right)$

$= 64\ \text{cis}(-9\pi)$

$= 64\ \text{cis}\ \pi$ (2)

$= -64$

(b) $|z - i| = |z + 3|$



$m_{AB} = \frac{1}{3}$

$m_l = -3$

$\therefore$ Locus is $y - \frac{1}{2} = -3(x + \frac{3}{2})$

$y - \frac{1}{2} = -3x - \frac{9}{2}$

$y = -3x - 4$ (2)

(c) $Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$ $(z \neq 0)$

$Re\left(\frac{1}{x+iy}\right) \leq \frac{1}{2}$

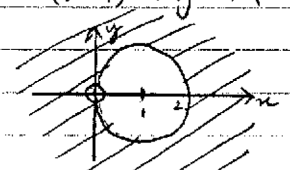
$Re\left(\frac{x - iy}{x^2+y^2}\right) \leq \frac{1}{2}$

$\therefore \frac{x}{x^2+y^2} \leq \frac{1}{2}$

$\therefore 2x \leq x^2 + y^2$

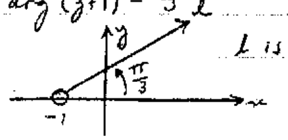
$\therefore x^2 - 2x + 1 + y^2 \geq 1$

$\therefore (x-1)^2 + y^2 \geq 1$ $(z \neq 0)$



(3)

(d) (i) $\arg(z+1) = \frac{\pi}{3}$ l



l is $y - 0 = \tan\frac{\pi}{3}(x+1)$

$y = \sqrt{3}(x+1)$

(1)

(ii) $|z|$ is a minimum at A where $OA \perp l$.

A is $(a, \sqrt{3}(a+1))$

$m_{OA} = \frac{\sqrt{3}(a+1)}{a} = -\frac{1}{\sqrt{3}}$

$\therefore 3(a+1) = -a$

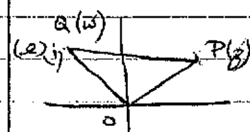
$\therefore 4a + 3 = 0$

$\therefore a = -\frac{3}{4}$ (2)

$\therefore$ A is $\left(-\frac{3}{4}, \frac{\sqrt{3}}{4}\right)$

i.e. $z = -\frac{3}{4} + i\frac{\sqrt{3}}{4}$ is the required solution

(e) (i)



$w = z\ \text{cis}\ \frac{\pi}{3}$

as $|w| = |z|$ and

$\angle QOP = \frac{\pi}{3}$ (equilateral $\triangle$)

$\therefore wz = \left(z\ \text{cis}\ \frac{\pi}{3}\right).z$

$= z^2\ \text{cis}\ \frac{\pi}{3}$ (1)

(ii) $z^2 + w^2 = z^2 + z^2\ \text{cis}\ \frac{2\pi}{3}$

$= z^2\left(1 + \text{cis}\ \frac{2\pi}{3}\right)$

$= z^2\left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= z^2\left(1 - \frac{1}{2} + i.\frac{\sqrt{3}}{2}\right)$

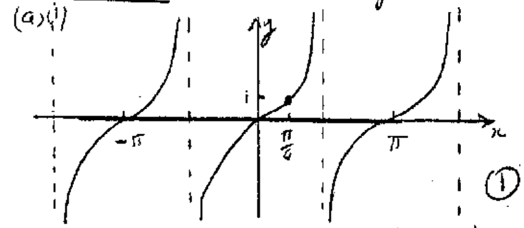
$= z^2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= z^2\ \text{cis}\ \frac{\pi}{3}$ (2)

$= wz$

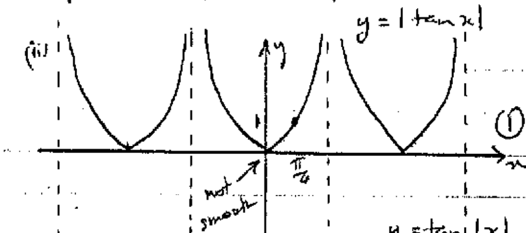
Question 3

(a)(i)



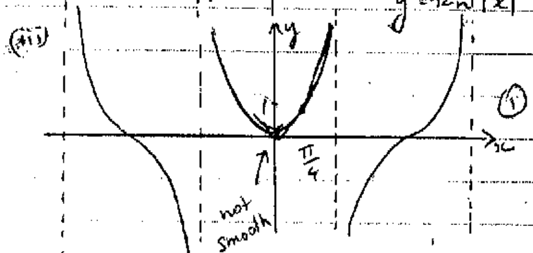
①

(ii)



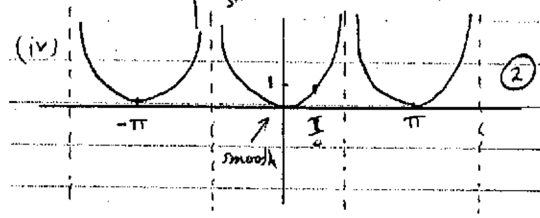
①

(iii)



①

(iv)



②

(b) $f(x) = \dfrac{x}{\ln x}, \quad x > 0$

(i) Domain $0 < x < 1, \; x > 1$

Asymptote $x = 1$ ②

(ii) $f'(x) = \dfrac{\ln x . 1 - x . \frac{1}{x}}{(\ln x)^2}$

$= \dfrac{\ln x - 1}{(\ln x)^2}$

$f''(x) = \dfrac{(\ln x)^2 . \frac{1}{x} - (\ln x - 1) . 2\ln x . \frac{1}{x}}{(\ln x)^4}$

$= \dfrac{\ln x - 2\ln x + 2}{x(\ln x)^3}$

$= \dfrac{2 - \ln x}{x(\ln x)^3}$

For turning point: $\ln x - 1 = 0$

$\therefore \ln x = 1$

$\therefore x = e$

$\therefore y = \dfrac{e}{\ln e} = e$

$y'' = \dfrac{2 - \ln e}{e . (\ln e)^3}$

$= \dfrac{2-1}{e . 1}$

$= \dfrac{1}{e} > 0$ ③

$\therefore$ Min turning point at (e, e)

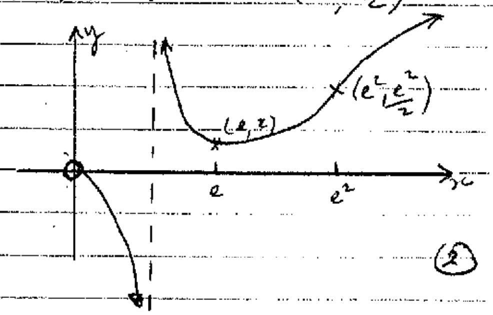
(iii)

x	$e^2 - \varepsilon$	e^2	$e^2 + \varepsilon$
y''	$\dfrac{2 - \ln(e^2 - \varepsilon)}{(e^2-\varepsilon)\ln(e^2-\varepsilon)}$ $= \dfrac{+ve}{+ve . +ve}$ $= +ve$	$\dfrac{2 - \ln e^2}{e^2(\ln e^2)^3}$ $= 0$	$\dfrac{2 - \ln(e^2+\varepsilon)}{(e^2+\varepsilon)\ln(e^2+\varepsilon)}$ $= \dfrac{-ve}{+ve . +ve}$ $= -ve$
Concavity	up	—	down

③

$\therefore$ Change of concavity at $x = e^2$

$\therefore$ Pt of inflexion at $\left(e^2, \dfrac{e^2}{2}\right)$



②

[2] 4. (a) (i)

$$\begin{aligned} z^3 - 1 &= 0, \\ (z-1)(z^2+z+1) &= 0, \\ \therefore\ z &= 1 \text{ or } \frac{-1 \pm \sqrt{1-4}}{2}, \\ &= 1 \text{ or } \frac{-1 \pm \sqrt{3}i}{2}. \end{aligned}$$

[1] (ii) $(\omega - 1)(\omega^2 + \omega + 1) = 0$ from (i).
Now $\omega \neq 1$, $\therefore\ \omega^2 + \omega + 1 = 0$.

[3] (iii)

$$\begin{aligned} \alpha + \beta &= 4 + \omega + 4 + \omega^2, \\ &= 7 + 1 + \omega + \omega^2, \\ &= 7. \\ \alpha\beta &= 16 + 4\omega + 4\omega^2 + \omega^3, \\ &= 12 + 4(1 + \omega + \omega^2) + 1, \\ &= 13. \end{aligned}$$

$\therefore\ z^2 - 7z + 13 = 0.$

[3] (b)

$$\begin{aligned} P(x) &= x^3 + ax^2 + bx + c. \\ P(0) &= c = 4. \\ P(x) &= (x^2+4)(x+\alpha) + x + 8. \\ P(0) &= 4\alpha + 8 = -4, \\ \alpha &= -3. \end{aligned}$$

$(x^2+4)(x-3) + 8 = x^3 - 3x^2 + 4x - 12 + x + 8.$
$\therefore\ P(x) = x^3 - 3x^2 + 5x - 4.$

[1] (c) (i) As the polynomial has real coefficients, if $(z - i\theta)$ is a factor then $(z + i\theta)$ is also a factor (conjugate root theorem).
i.e., $P(-i\theta) = 0$.

[1] (ii) $(z^2 + \theta^2)$ is a factor of $P(x)$. Let $(z - \alpha)$ be the last factor.
Sum of roots is $\theta i - \theta i + \alpha = -a$.
i.e., $\alpha = -a$ which is real,
$\therefore\ \alpha$ is real and there is one real root.

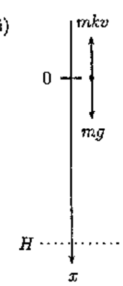
[4] (iii) Taking roots two at a time,

$$\begin{aligned} b &= \theta^2 + \theta i a - \theta i a, \\ &= \theta^2. \end{aligned}$$

$\therefore\ b > 0$ as $\theta \in \mathbb{R}$.

Product of roots,
$$\begin{aligned} -c &= -\theta^2 i^2(-a), \\ c &= \theta^2 a, \\ &= ab. \end{aligned}$$

[1] 5. (a) (i)



$$\begin{aligned} \sum F_x &= ma = mg - mkv, \\ a &= g - kv. \\ \textit{i.e.}\quad v\frac{dv}{dx} &= g - kv. \end{aligned}$$

4 (ii) $\int dx = \int \frac{v\,dv}{g-kv}$,

$$= -\frac{1}{k}\int \frac{g-kv}{g-kv}dv + \frac{-g}{k^2}\int \frac{-k\,dv}{g-kv}.$$

$$x = -\frac{v}{k} - \frac{g}{k^2}\ln(g-kv) + c.$$

When $x=0$, $v=0$, $\therefore\ c = \frac{g}{k^2}\ln g$.

$$x = \frac{g}{k^2}\ln\left(\frac{g}{g-kv}\right) - \frac{v}{k}.$$

When $x = H$, $v = V_0$,

$$H = \frac{g}{k^2}\ln\left(\frac{g}{g-kV_0}\right) - \frac{V_0}{k}.$$

Rearranging, $\ln\left(\frac{g-kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0$,

i.e., $\ln\left(1-\frac{kV_0}{g}\right) + \frac{kV_0}{g} + \frac{k^2H}{g} = 0$.

3 (iii) $\frac{dv}{dt} = g - kv$,

$$\int dt = \frac{-1}{k}\int \frac{-k\,dv}{g-kv},$$

$$t = -\frac{1}{k}\ln(g-kv) + c.$$

When $t=0$, $v=0$, $\therefore\ c = \frac{1}{k}\ln g$,

So $t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$.

When $t = T$, $v = V_0$,

$\therefore\ T = \frac{1}{k}\ln\left(\frac{g}{g-kV_0}\right)$.

2 (iv) $\ln\left(1-\frac{kV_0}{g}\right) = -kT$ from (iii).

Substitute in (ii),

$$-kT + \frac{kV_0}{g} + \frac{k^2H}{g} = 0,$$

$$\frac{kV_0}{g} = kT - \frac{k^2H}{g},$$

$$V_0 = gT - kH.$$

1 (v) Terminal velocity occurs when $\ddot{x} = 0$,

i.e. $V_T = \frac{g}{k}$.

Now $V_0 < V_T$,

$\therefore\ V_0 < \frac{g}{k}$

$$T = \frac{V_0}{g} + \frac{kH}{g} \text{ from (iv)},$$

$$T < \frac{g}{k}\times\frac{1}{g} + \frac{kH}{g},$$

$$T < \frac{1}{k} + \frac{kH}{g}.$$

2 (b) At least two together $\Longrightarrow$ not all separate.
Total number of arrangements in a circle $= 7!$
Number of arrangements where separated $= 3!4!$
$\therefore$ Ways with at least two together $= 7! - 3!4!$
$= 4896.$

2 (c) Number of ways $= \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5},$
$= 2^5 - 1,$
$= 31.$

2 6. (a) (i) $\text{cis}\,3\theta = (\text{cis}\,\theta)^3$, by De Moivre's Theorem.
i.e., $\cos 3\theta + i \sin 3\theta = \cos^3\theta + 3i \sin\theta \cos^2\theta + 3i^2 \sin^2\theta \cos\theta + i^3 \sin^3\theta.$
Equating real and imaginary parts,

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\sin^2\theta\cos\theta,\\
&= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta,\\
&= \cos^3\theta - 3\cos\theta + 3\cos^3\theta,\\
&= 4\cos^3\theta - 3\cos\theta.\\
\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta,\\
&= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta,\\
&= 3\sin\theta - 3\sin^3\theta - \sin^3\theta,\\
&= 3\sin\theta - 4\sin^3\theta.
\end{aligned}$$

2 (ii)

$$\begin{aligned}
\cot 3\theta &= \frac{\cos 3\theta}{\sin 3\theta},\\
&= \frac{4\cos^3\theta - 3\cos\theta}{3\sin\theta - 4\sin^3\theta},\\
&= \frac{4\cot^3\theta - 3\cot\theta.\sec^2\theta}{3\sec^2\theta - 4},\\
&= \frac{4\cot^3\theta - 3\cot\theta.(1+\cot^2\theta)}{3(1+\cot^2\theta) - 4},\\
&= \frac{4\cot^3\theta - 3\cot\theta - 3\cot^3\theta}{3 + 3\cot^2\theta - 4},\\
&= \frac{\cot^3\theta - 3\cot\theta)}{3\cot^2\theta - 1},\\
&= \frac{t^3 - 3t}{3t^2 - 1}, \text{ using } t = \cot\theta.
\end{aligned}$$

2 (iii) Now $\cot 3\theta = 1,$ $\qquad 0 \le \theta \le 2\pi$

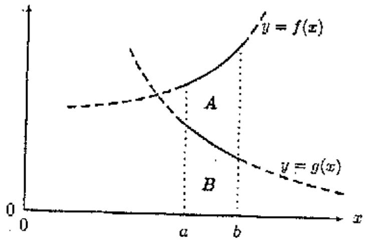
$3\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}$ $\qquad 0 \le 3\theta \le 6\pi \ \left(\frac{24\pi}{4}\right)$

$\therefore\ \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{21\pi}{12}.$

2 (iv) $\frac{t^4 - 3t}{3t^2 - 1} = 1.$
$\therefore\ t^3 - 3t^2 - 3t + 1 = 0.$
As $\cot\theta = \cot(\pi + \theta),$
$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}$ are the only distinct values from (iii) above.
So $t = \cot\theta = \cot\frac{\pi}{12},\ \cot\frac{5\pi}{12},\ \cot\frac{9\pi}{12}$ are the roots.
Product of the roots, $-1 = \cot\frac{\pi}{12}.\cot\frac{5\pi}{12}.\cot\frac{9\pi}{12}.$

[1] (v) $\dfrac{1}{x^3} - \dfrac{3}{x^2} - \dfrac{3}{x} + 1 = 0,$
$x^3 - 3x^2 - 3x + 1 = 0.$

[2] (b) (i)



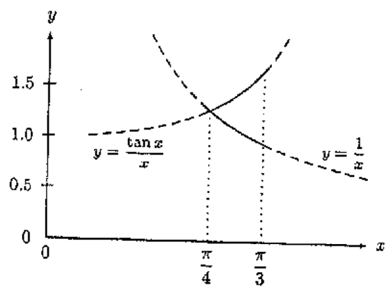
$\int_a^b f(x)\,dx$ is shown by $A + B$ and
$\int_a^b g(x)\,dx$ is shown by B.
It is clear that $A + B > B$,
i.e., $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$.

[1] (ii) $y = \tan x,$
$y' = \sec^2 x > 1 \;\forall\, x.$
$\therefore$ $\tan x$ is an increasing function, $\frac{\pi}{4} \le x \le \frac{\pi}{3}$ (discontinuities at $\pm\frac{\pi}{2}$ are outside the range).

[3] (iii) When $x = \frac{\pi}{4}$, $\tan x = 1$,
and for $\frac{\pi}{4} < x \le \frac{\pi}{3}$, $\tan x > 1$ as $\tan x$ is an increasing function.
$\therefore$ $\dfrac{\tan x}{x} > \dfrac{1}{x}$ as $x > 0$.
$\therefore$ by part (i): $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}dx > \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{x}.$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{x} = [\ln x]_{\frac{\pi}{4}}^{\frac{\pi}{3}},$$
$$= \ln\left(\tfrac{\pi}{3}\cdot\tfrac{4}{\pi}\right),$$
$$= \ln\frac{4}{3}.$$

i.e., $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\tan x}{x}dx > \ln\frac{4}{3}$. [See sketch.]



QUESTION 7

(a) (i) $I_n = \int_1^e x(\ln x)^n \, dx$

$= \int_1^e \frac{d}{dx}\left(\frac{1}{2}x^2\right).(\ln x)^n \, dx.$

$= \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \int_1^e \frac{1}{2}x^2 . \, n(\ln x)^{n-1} \frac{1}{x} . \, dx.$

$= \frac{1}{2}e^2(\ln e)^n - \frac{1}{2}.1^2(\ln 1)^n - \frac{n}{2}\int_1^e x \, (\ln x)^{n-1} \, dx$

$= \frac{1}{2}e^2 - \frac{1}{2}.0 - \frac{n}{2} I_{n-1}$

$\therefore \; I_n = \frac{1}{2}e^2 - \frac{n}{2} I_{n-1}.$

(ii) $I_3 = \frac{e^2}{2} - \frac{3}{2} I_2$

$= \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{2} - I_1\right]$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}\left[\frac{e^2}{2} - \frac{1}{2} I_0\right]$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4} I_0 \quad ; \quad I_0 = \int_1^e x \, dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}.$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8}$

$\therefore \; I_3 = \frac{e^2}{8} + \frac{3}{8}.$

(b) $V = \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{6}} 2\pi x y \, \delta x.$

$= \int_0^{\sqrt{6}} 2\pi x y \, dx.$

$= 2\pi \int_0^{\sqrt{6}} x^3(6 - x^2) \, dx.$

$= 2\pi \int_0^{\sqrt{6}} (6x^3 - x^5) \, dx$

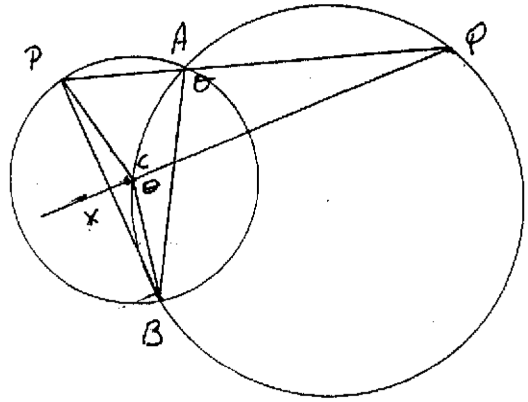
$= 2\pi \left[\frac{3x^4}{2} - \frac{x^6}{6}\right]_0^{\sqrt{6}}$

$= 2\pi \left[54 - 36\right]$

$= 36\pi \; u^3$

Q7 (contd)

(c)



(ii) now $B\hat{C}Q = B\hat{A}Q = \theta$ (angles in the same segment standing on the same arc, are equal)

$\therefore$ $B\hat{C}X = 180° - \theta$ (angles are supplementary)

(iii) now $P\hat{A}B = 180° - \theta$ (supplementary angles)

$P\hat{C}B = 360° - 2\theta$ (angle at the centre is double the angle at the circumference standing on same arc)

$\therefore$ $P\hat{C}X = X\hat{C}B$ $(180° - \theta + 180° - \theta = 360° - 2\theta)$

CX is common

PC = BC (equal radii)

$\therefore$ $\Delta PCX \equiv \Delta BCX$ (SAS)

$\therefore$ $P\hat{X}C = C\hat{X}B$ (corresponding angles of congruent Δ's)

now $P\hat{X}C + C\hat{X}B = 180°$ (supplementary angles)

$\therefore$ $P\hat{X}C = 90°$.

QUESTION 2.

(a) The roots of $x^3+ax^2+bx+c=0$ ④ are α, β & γ (NB must have 3 roots)

need to establish equation with roots $\alpha\beta, \alpha\gamma$ & $\beta\gamma$.

now $\alpha\beta\gamma = -c$.

$\alpha\beta = \frac{-c}{\gamma}$ ∴ Use $X = \frac{-c}{x}$ ie $x = -\frac{c}{X}$ in ④

$\left(\frac{-c}{X}\right)^3 + a\left(\frac{-c}{X}\right)^2 + b\left(\frac{-c}{X}\right) + c = 0$

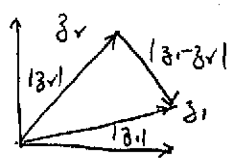
$\frac{-c^3}{X^3} + \frac{ac^2}{X^2} - \frac{bc}{X} + c = 0$

$-c^3 + ac^2X - bcX^2 + cX^3 = 0$

$\Rightarrow X^3 - bX^2 + acX - c^2 = 0$

OR $x^3 - bx^2 + acx - c^2 = 0$

(b) (i)



From triangle inequality, "the sum of the lengths of any two sides exceeds the third side".

ie $|z_1 - z_2| + |z_2| \geq |z_1|$

∴ $|z_2 - z_1| \geq |z_1| - |z_2|$.

(ii) Since $\left|z - \frac{4}{z}\right| = 2$ and $\left|z - \frac{4}{z}\right| \geq |z| - \left|\frac{4}{z}\right|$

then $|z| - \frac{4}{|z|} \leq 2$.

$|z|^2 - 4 \leq 2|z|$

$|z|^2 - 2|z| - 4 \leq 0$

$|z|^2 - 2|z| + 1 \leq 5$

$(|z| - 1)^2 \leq 5$

∴ $-\sqrt{5} \leq |z| - 1 \leq \sqrt{5}$

$-\sqrt{5} + 1 \leq |z| \leq \sqrt{5} + 1$

∴ $|z| \leq \sqrt{5} + 1 \Rightarrow$ max. value of $|z| = \sqrt{5} + 1$.

8. (c) (i) If $P(x)=0$ has a root of multiplicity n, say α.

then $P(x)=(x-\alpha)^n \cdot Q(x)$

$$P'(x) = n(x-\alpha)^{n-1}Q(x) + (x-\alpha)^n \cdot Q'(x).$$

$$= (x-\alpha)^{n-1}\left[n\,Q(x) + (x-\alpha)\,Q'(x)\right]$$

now since $Q'(x)$ is a polynomial

$n\,Q(x) + (x-\alpha)\,Q'(x)$ is a polynomial say $T(x)$.

$\therefore P'(x) = (x-\alpha)^{n-1}\,T(x).$

which has a root α of multiplicity $n-1$.

(ii) given $x^3+3px^2+3qx+r=0$ has a multiple root. (say α)

$\therefore \alpha^3+3p\alpha^2+3q\alpha+r=0$ — (1)

& $3\alpha^2+6p\alpha+3q=0$ (2) from (i).

$\therefore \alpha^2+2p\alpha+q=0.$ (2a)

hence $\alpha^3+2p\alpha^2+q\alpha=0.$ — (3).

From (1) & (3)

$p\alpha^2+2q\alpha+r=0$ — (4)

(2a) $\times p$

$p\alpha^2+2p^2\alpha+pq=0$ (2b)

2(b) − (4)

$2(p^2-q)\alpha+pq-r=0$

$\therefore 2(p^2-q)\alpha = r-pq$

$$\boxed{\alpha = \frac{r-pq}{2(p^2-q)}}$$